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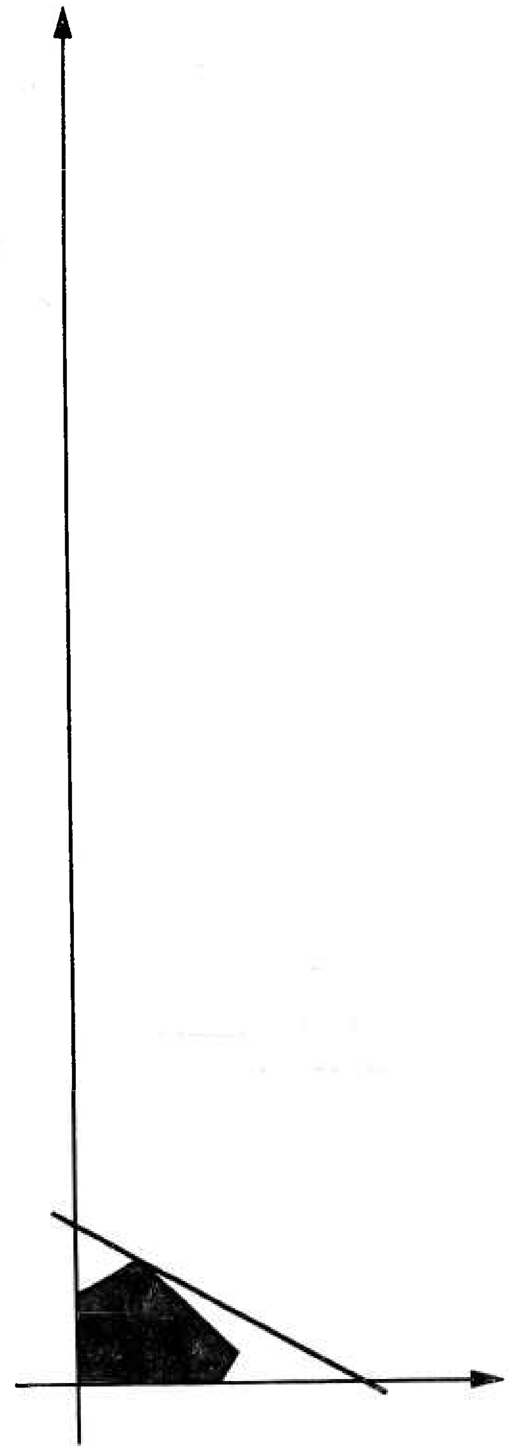
J. Abadie



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ABSTRACT

If, in applying the Decomposition Principle of George B. Dantzig and Philip Wolfe, [1], [2], we start from a feasible basic solution, say x^o, we obtain at the end of the first iteration, a feasible solution, say x^1, which generally is not a basic one. However, one can easily compute, from x^o and x^1, the feasible solution, say $\bar{x}$, which, on the half-line $x^o + \theta (x^1 - x^o)$, $\theta \geq 0$, is the farthest from x^o. The question then arises: to find an expeditious way for starting the next iteration with $\bar{x}$ (so as to have Decomposition Principle altogether with feasible basic solutions).

In this report, we try some possible ways along that line. The conclusions are:

1. The best of the algorithms which are examined is that of Section 4 and 5;
2. This latter algorithm is shown, in Section 6, to be the simplex algorithm in the case where it is kept, in the memories of the computer, some special kind of a combination of the original data and of some currently computed quantities;
3. Finally it is shown that there exists very little differences between this algorithm and the algorithm of G. B. Dantzig, A. S. Manne, W. Orchard-Hays, and J. T. Robacher [3], and that it is possible to find many other algorithms along the same line of ideas.

ON DECOMPOSITION PRINCIPLE

1. Introduction and Notations

We consider the following linear program:

Program 1 — Minimize cx

(1) subject to $Ax = a$

(2) $Bx = b$

(3) $x \geq 0$.

We assume that x is any point in the n-dimensional real vector space R^n (i.e., any column vector with n real elements) that A is a $p \times n$ matrix, that B is $q \times n$ matrix, and that the $p + q$ equations (1), (2) are independent; we set $p + q = m$.

We shall call Q the polyhedron

(2) $Bx = b$

(3) $x \geq 0$

and M the polyhedron defined by (1), (2), (3); in other words, Q is the section of M by the linear variety $Ax = a$.

We shall assume, throughout this report, that there is no degeneracy of any kind (that is, we disregard any difficulty which can be avoided by some ε-perturbation of the right-hand sides a , b or some ε-perturbation of the cost-vector c; we even assume that these perturbations, if necessary, have been made previously).

Let X^o be an extreme point (a vertex) of Q ; we set

$$J = J(X^o) = \{j : X^o_j > 0\}$$

Then, by the non-degeneracy assumption, the matrix whose columns are $(B^j)_{j \in J}$ (and which we call B^J) is a square nonsingular matrix.

Multiplying on the left (2) by its inverse matrix, say β_J we have an equation equivalent to (2) of the shape:

$$(4) \qquad x_J + \tilde{B}^{\bar{J}} x_{\bar{J}} = \tilde{b} ,$$

where;

x_J is the q-dimensional vector $(x_j)_{j \in J}$;

$x_{\bar{J}}$ is the (n-q) - dimensional vector $(x_j)_{j \in \bar{J}}$

and $\bar{J}$ is the complement of J (that is: $x_{\bar{J}}$ is the set of components of x which are not component of x_J);

$\tilde{B}$ is the product matrix $\beta_J B$, and $\tilde{B}^{\bar{J}}$ is the submatrix of $\tilde{B}$ composed with the columns $(B^j)_{j \in \bar{J}}$;

$\tilde{b}$ is q-dimensional vector $\beta_J b$, that is X^o_J (the q-dimensional vector whose components are $(X^o_j)_{j \in J}$) .

We shall say that <u>we express the polyhedron Q with the extreme point X^o</u> for saying that we replace (2) by (4) .

Corresponding to any extreme point X^o , there is a cone, say $C(X^o)$, which is defined by

$$(5) \qquad By = 0$$

$$(6) \qquad y_{\bar{J}} \geq 0 \quad \text{and} \quad y_J \text{ (unrestricted)}$$

This cone $C(X^o)$ is characterized by the following property: if $y \in C(X^o)$, then $X^o + \theta y \in C(X^o)$ for some strictly positive scalar θ . We shall write an explicit formula for points in $C(X^o)$: let Q be expressed with X^o ; then the extreme half-lines of $C(X^o)$ are generated by the following vectors Y^j , where j is any index in $\bar{J}$ (so

that $C(X^o)$ has exactly $n - q$ extreme half-lines): the components Y_k^j of Y^j are

(7) $$Y_j^j = 1$$

(8) $$Y_k^j = -\tilde{B}_k^j \,, \quad \text{if } k \in J$$

(9) $$Y_k^j = 0 \,, \quad \text{if } k \in \overline{j+J}$$

(where $J+j$ means the union of the two sets J, $\{j\}$) and the bar denotes the complementary set. From (7) and (9) these $n-q$ vectors are independent (each one has a component equal to 1, while this same component is zero for any other).

The explicit formula for points in $C(X^o)$ is

(10) $$y = \sum_{j \in \bar{J}} Y^j \mu_j$$

where the μ_j's are nonnegative arbitrary scalars.

2. One Iteration of the Decomposition Principle, Starting from an Extreme Point of M.

In this section, we assume that the polyhedra Q and M are both bounded. The general case could easily be covered, at the expense of some more explanations.

Let x^o be an extreme point of M; then by the non-degeneracy assumption, x^o is in the relative interior of a p-dimensional simplex, say $F^p(x^o)$, of Q, and so can be expressed as a convex combination of $p+1$ extreme points, say $X^o, X^1, \ldots, X^p$, of the face $F^p(x^o)$ (that is to say: $p+1$ extreme points of Q which belong to $F^p(x^o)$; they form a p-dimensional simplex). Let this expression be

$$(1)\quad x^o = \sum_{\ell=o}^{p} X^{\ell}\lambda_{\ell}^{o}$$

where the scalars λ_{ℓ}^{o} satisfy

$$(2)\quad \lambda_{\ell}^{o} > 0 \qquad \ell = 0,1,\ldots,p$$

$$(3)\quad \sum_{\ell=0}^{p} \lambda_{\ell}^{o} = 1$$

(the case where one or more λ_{ℓ}^{o} , $\ell = 0,1,\ldots,p$ is zero corresponds to some degeneracy which is excluded by assumption) We can equivalently express x^o as

$$(4)\quad x^o = X^o + \sum_{\ell\in L} (X^{\ell} - X^o)\ \lambda_{\ell}^{o}$$

where $L = \{1,2,\ldots,p\}$ and where the scalars $(\lambda_{\ell}^{o})_{\ell\in L}$ satisfy

$$(5)\quad \lambda_{\ell}^{o} > 0, \forall\ \ell\in L$$

$$(6)\quad \sum_{\ell\in L} \lambda_{\ell}^{o} < 1 ,$$

We assume that Q is expressed with the extreme point X^o , and we consider $C(X^o)$. Among its extreme half-lines, some may be in $F^p(x^o)$, and some are not. We set

$$(7)\quad K = J(x^o) = \{j: x_j^o > 0\}$$

(8) $I = J(x^o) - J(X^o) = K-J$ (the relative complement of J with respect to K). Then $F^p(x^o)$ is defined by

$$(9)\quad x_J + \tilde{B}^I\ x_I = \tilde{b}$$

$$(10)\qquad x_I \geq 0$$

(11) $\qquad x_J \geq 0$

(12) $\qquad x_{\overline{K}} = 0$,

where the vectors x_I , $x_{\overline{K}}$ are defined by

$$x_I = (x_j)_{j \in I}$$

$$x_{\overline{K}} = (x_j)_{j \in \overline{K}} ,$$

and where $\tilde{B}^I$ is the submatrix of $\tilde{B}$ where colums are $(B^j)_{j\in I}$. Now, the extreme half-lines of $C(X^o)$ which are not in $F^p(x^o)$ are

$$X = X^o + \theta Y^s$$

where the scalar θ is ≥ 0 , and the index $s \in \overline{K}$ (notice that $\overline{K}$ is not empty if we assume, as we do, that x^o is not the unique solution of $Ax = a$, $Bx = b$ and $x \geq 0$). Since Q is bounded, there is a maximum value of θ such that $X \geq 0$: let this maximum value be θ_s . Then, we set

$$X^{p+1} = X^o + Y^s \theta_s ,$$

and we notice that X^{p+1} is an extreme point of Q .

We consider the (p+1)-dimensional simplex whose p+2 extreme points are $X^o, X^1, X^2, \ldots, X^{p+1}$; one of its p-dimensional simplexes is precisely $F^p(x^o)$.

We take the intersection of this (p+1)-dimensional simplex with the (n-p)-dimensional linear variety $Ax = a$. This intersection exists (x^o is one point of this intersection). Moreover, since x^o belongs to the relative interior of $F^p(x^o)$, this intersection is a segment $[x^o,x^1]$, where x^1 belongs to the relative interior of one of the p-dimensional simplexes of the (p+1)-dimensional simplex. This p-dimensional simplex containing x^1 is distinct from $F^p(x^o)$. Then x^1 can be expressed as a convex combination of p+1

points among the p+2 points $X^o, X^1, X^2, \ldots, X^{p+1}$; moreover, X^{p+1} must be taken (with strictly positive coefficient) in this combination, and one of the p+1 points $X^o, X^1, X^2, \ldots, X^p$ must have zero coefficient.

Case 1, X^o has a zero coefficient in the convex expression of x^1: we then take X^{p+1} (for expressing Q) instead of X^o ("change of basis in the sub-problem").

Case 2, X^o has a strictly positive coefficient in the expression of x^1: there is no need to change the expression of Q (no change of basis in the sub-problem).

The two cases are illustrated by the figures 1 (for Case 1) and 2 (for Case 2), where p = 1 and the pentahedra represent $F^p(x^o)$.

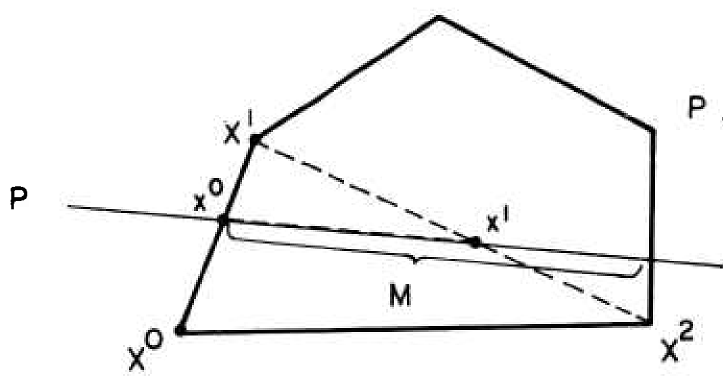

FIG. 1

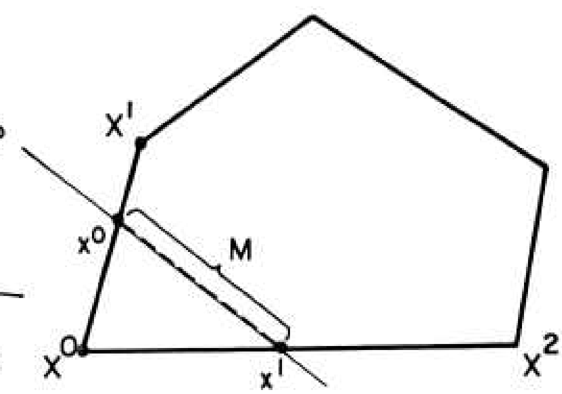

FIG. 2

We see that, in Case 1, x^1 is no more an extreme point of M (while it is in Case 2).

In order to make a step toward the solution of Program 1, we need $cx^1 < cx^o$. Some results now will be useful; the first two of them do not need any proof

The hypothesis, when not expressed, are those above in the present and first sections.

<u>LEMMA 1</u> - The set of vectors $(X^\ell - X^o)_{\ell \in L}$ is an independent set.

<u>LEMMA 2</u> - (Corollary of Lemma 1). The matrix whose columns are $A(X^\ell - X^o)$ $\ell \in L$ is a square and nonsingular matrix.

We denote by α the square matrix which is the inverse of the matrix in Lemma 2.

<u>THEOREM 1</u> - The hyperplane containing $F^p(x^o)$ and parallel to $Ax = 0$ and to $cx = 0$ has, as an equation:

$$(13) \qquad (c - u_o A)\, x = (c - u_o A)\, X^o ,$$

where u_o is the unique solution of

$$(14) \qquad (c - uA)\,(X^\ell - X^o) = 0 , \quad \forall \ell \in L.$$

<u>PROOF</u> - It is well known that the general equation of a hyperplane parallel to $Ax = 0$ and to $cx = 0$ is

$$(vc - uA)x = w ,$$

where v , w are scalars, and u is a row-vector. Moreover, the hyperplane (13) contains X^o , and contains X^ℓ if and only if (14) is true. But, by lemma 2, (14) has one and only one solution, say u_o . The hyperplane (13) contains $p+1$ points X^o , $X^1,\dots,X^p$ of the p-dimensional face $F^p(x^o)$: it then entirely contains this face. Finally, the proof shows the uniqueness of the hyperplane, which justifies the word "the" at the beginning of

theorem 1.

THEOREM 2 - We have

(15) $$cx^1 < cx^o$$

if and only if

(16) $$(c - u_oA)\, X^{p+1} < (c - u_oA)\, X^o$$

PROOF - We first notice that

(17) $$cx^1 - cx^o = (c - u_oA)x^1 - (c - u_oA)x^o ,$$

since u_oAx^1 and u_oAx^o are both equal to u_oa.

Moreover, x^1 belongs to the $(p+1)$-dimensional simplex generated by $X^o , X^1,\ldots,X^{p+1}$

In Case 1, x^1 belongs to the p-dimensional simplex generated by $X^1 , X^2,\ldots,X^{p+1}$. We then have by formula (4):

(18) $$x^o = X^o + \sum_{\ell \in L} (X^\ell - X^o)\, \lambda_\ell^o ,$$

and by a similar argument:

(19) $$x^1 = X^{p+1} + \sum_{\ell \in L} (X^\ell - X^{p+1})\, \lambda_\ell^1 ,$$

where

(20) $$\lambda_\ell^1 > 0 , \quad \forall\, \ell \in L$$

and

(21) $$\sum_{\ell \in L} \lambda_\ell^1 < 1 .$$

From (17), (18), (19) and (14), we have

$$cx^1 - cx^o = (c - u_o A)\,(X^{p+1} - X^o) + \sum_{\ell \in L} (c - u_o A)(X^\ell - X^{p+1})\,\lambda_\ell^1 .$$

Whence, replacing

$$X^\ell - X^{p+1} \quad \text{by} \quad X^\ell - X^o + X^o - X^{p+1} ,$$

$$cx^1 - cx^o = (1 - \sum_{\ell \in L} \lambda_\ell^1)\,(c - u_o A)\,(X^{p+1} - X^o) ,$$

which, due to (21) proves the theorem in Case 1.

In Case 2, x^1 belongs to a p-dimensional simplex generated (by X^o, X^{p+1} and p-1 among the p vectors X^1, $X^2,\ldots,X^p$,) say by X^o, X^2, $X^3,\ldots,X^p$, X^{p+1}. Then,

$$x^1 = X^o + \sum_{\ell \in L} (X^\ell - X^o)\,\lambda_\ell^1 + (X^{p+1} - X^o)\,\lambda_{p+1}^1 ,$$

where $\lambda_1^1 = 0$

$$\lambda_\ell^1 > 0, \quad \ell = 2, 3,\ldots, p, p+1$$

$$\sum_{\ell=1}^{p+1} \lambda_\ell^1 < 1 .$$

We have then, by the same computation (somewhat simpler) as in Case 1:

$$cx^1 - cx^o = \lambda_{p+1}^1 (c - u_o A)\,(X^{p+1} - X^o)$$

which ends the proof in Case 2.

In practice, the computation of x^1 needs the following steps:

Step 1, compute u_o, as uniquely given by

$$u_o A(X^\ell - X^o) = c(X^\ell - X^o) , \ \forall \ell \in L .$$

Step 2, express the "adjusted cost row" $c - u_o A$ with the extreme point X^o, of the polyhedron Q, that is to say, compute for any $k \in \bar{J}$:

$$\delta^k = c^k - u_o A^k - \sum_{j \in J} (c^j - u_o A^j) \bar{B}_j^k ;$$

2.1, if $\delta^k \geq 0$, $k \in \bar{J}$, then x^o is an optimal solution (theorem 2);

2.2, let $s \in \bar{J}$ be such that $\delta^s < 0$; consider Y^s (as defined in section 1), and compute X^{p+1} by

$$X^{p+1} = X^o + \Theta_s Y^s \geq 0$$

$$\Theta_s = \min_{j \in J} \left\{ \frac{X_j^o}{-Y_j^s} \mid Y_j^s < 0 \right\}$$

(Θ_s exists, since Q is bounded, and is strictly positive, by the nondegeneracy assumption).

Step 3, Compute the intersection of $Ax = a$ with the (p+1)-dimensional simplex spanned (by means of convex combinations) by $X^o, X^1, X^2, \dots, X^{p+1}$.

This intersection is a segment (of which x^o is an extremity), any point of which can be expressed as

$$x = \sum_{\ell=0}^{p+1} X^\ell (\lambda_\ell^o + \mu_\ell) , \qquad \text{with } \lambda_{p+1}^o = 0$$

where

$$\sum_{\ell=0}^{p+1} \mu_\ell = 0 \text{ and } \lambda_\ell^o + \mu_\ell \geq 0 .$$

[Note that $\sum_{\ell=0}^{p+1} (\lambda_\ell^o + \mu_\ell) = 1$ and μ_ℓ (unrestricted)].

Also x should satisfy

$$A(x-x^{o}) = 0 \quad \text{and} \sum_{\ell=0}^{p+1} \mu_\ell = 0 \ .$$

We have:

$$\sum_{\ell=1}^{p} A(X^\ell - X^{o})\mu_\ell + A(X^{p+1} - X^{o})\mu_{p+1} = 0 \ .$$

We have easily the μ_ℓ's $\ell = 1,\ldots,p+1$ as follows, since $X^{p+1} - X^{o} = Y^{s} \ \Theta_s$:

first, we compute ν_ℓ^{o} , $\ell = 1,2,\ldots,p$, as uniquely determined by

$$\sum_{\ell=1}^{p} (A(X^\ell - X^{o}) \ \nu_\ell^{o} + AY^{s} = 0 \ ;$$

then, we have the μ's proportional to the ν's (with $\nu_{p+1}^{o} = 1$).

Let y^{s} be defined by

$$y^{s} = Y^{s} + \sum_{\ell=1}^{p} (X^\ell - X^{o}) \ \nu_\ell^{o} \ .$$

Then the intersection of $Ax = a$ with the $(p+1)$-dimensional simplex spanned by X^{o} , $X^1,\ldots,X^{p+1}$ is:

$$x = x^{o} + \Theta y^{s} \ ,$$

where Θ is bounded by

$$\lambda_\ell^{o} + \Theta\nu_\ell^{o} \geq 0 \qquad , \quad \forall \ \ell \in L$$

that is to say, the maximum value of Θ is

$$\Theta_o = \operatorname*{Min}_{\ell \in L} \left\{ \frac{\lambda_\ell^{o}}{-\nu_\ell^{o}} \ \middle| \ \nu_\ell^{o} < 0 \right\}$$

(we must notice that we do have $\nu_\ell^o < 0$ for some ℓ, since

$$1 + \sum_{\ell=1}^{p} \nu_\ell^o = 0) .$$

In case 1 as well as in case 2, we have

$$x^1 = x^o + \theta_o y^s .$$

Step 4

4.1, if $\lambda_\ell^o + \theta_o \nu_\ell^o = 0$ for $\ell = 0$, then we are in case 1: perform the change of basis in the subproblem, from X^o to X^{p+1} ;

4.2 if $\lambda_\ell^o + \theta_o \nu_\ell^o > 0$ for $\ell = 0$, then no change of basis in the subproblem is necessary.

In order to compute u_o (in Step 1), and $(\lambda_\ell^o)_{\ell\in L}$ (in Step 3), it is necessary to know α, the inverse matrix of the matrix whose columns are $(A(X^\ell - X^o))_{\ell\in L}$. Assuming that we know α, we need one more computation step.

Step 5, In case 1, assume that we know the inverse matrix α of the matrix whose columns are $A(^\ell - X^o)$, $\ell\in L$; compute the inverse α' of the matrix whose columns are $A(X^\ell - X^{p+1})$, $\forall\ \ell\in L$. Then, we write

$$\begin{aligned} A(X^\ell - X^{p+1}) &= A(X^\ell - X^o) + A(X^o - X^{p+1}) , \\ &= A(X^\ell - X^o) - \theta_s AY^s . \end{aligned}$$

and we use the Morrison formula[5] (with a row vector $e = (1,1,1,\ldots 1)$ that is, $e^\ell = 1$, $\forall\ \ell\in L$):

$$\alpha' = \alpha + \tau\alpha\, AY^s\, e\, \alpha \qquad \text{where} \qquad \tau = \frac{\theta_s}{1-\theta_s e\alpha\, AY^s}$$

(τ exists, since we already know that α' exists).

In case 2, we have to compute the inverse matrix say α' of the matrix whose columns are

$$A(X^{\ell} - X^{o}) ,$$

where $\ell = 2,3,\ldots,p+1$.

Here, we notice that the columns of indices $\ell = 2,3,\ldots,p$ remain the same, while the column

$$A(X^{1} - X^{o})$$

is changed to

$$A(X^{p+1} - X^{o}) ,$$

which can be written as

$$A(X^{p+1} - X^{o}) = A(X^{1} - X^{o}) + A(X^{p+1} - X^{1}) ;$$

the same formula as above applies, <u>but</u> e must, here in case 2, be the vector whose components are all zeros, except one (with index 1), which has a unit value (that is: $e^{\ell} = 0$ if $\ell \in L$ but $\ell \neq 1$, $e^{1} = 1$). The iteration described above is completely equivalent to one iteration of the Decomposition Principle of G. B. Dantzig and P. Wolfe, when starting with an extreme point x^{o} of M: in the latter method, we have to solve systems whose matrix has, as its columns,

$$\begin{pmatrix} A\,X^{\ell} \\ 1 \end{pmatrix}$$

where $\ell = 0,1,2,\ldots,p$. It suffices to eliminate one unknown, chosen in the obvious manner, to have the matrix considered in our

description. We have already seen that x^1, thus obtained in one iteration of the subproblem in the Decomposition Principle, may be a non-extremal point of M, although x^o was. If we allow more than one iteration of the simplex method in the subproblem in the Decomposition Principle, then the same result applies. The reader will easily convince himself that, in the latter case, x^1 may even lie in the relative interior of M. The only real difference is that in the original Decomposition Principle we usually perform the change of basis from X^o to X^{p+1}, in Case 1 as well as in Case 2 (Contrary to Step 4, Case 1). The distinction between Case 1 and Case 2 has been introduced here for preparing the next sections; also the introduction of the matrix whose columns are $(A(X^{\ell} - X^o))_{\ell\in L}$, instead of the matrix $\left(A_1 X^{\ell}\right)_{\ell\in L}$ of the original method, has chiefly been made for the same purpose. Finally we notice that the 5 Steps Iteration explained above can be performed on a point x^o which is in M but is not an extreme point of M provided that this x^o is a convex combination of $p+1$ extreme points of Q which form a p-dimensional simplex.

3. Decomposition Principle with Extremal Points of M

In some numerical experiments, the Decomposition Principle has led to some lengthy computation in terms of the number of iterations. It has been suggested [4] that a potential reason for this is that the current point x^{ν} at the ν^{th} iteration is generally a (relatively) interior point of the polyhedron Q.

In this section again, for the sake of brevity, we shall

assume that Q (hence M also) is a bounded polyhedron, and we shall give some ways to have extreme points of M in the successive iterations.

3.1 Let again x^o be an extreme point of M. Once s, in Step 2 above, has been obtained, one can apply the Decomposition Principle to the Restricted Program obtained from Program 1 by setting $x_j = 0$, for any j except those in $J(x^o0$ and for $J = s$. Solving the Restricted Program would be the analogous for the Decomposition Principle, of <u>one</u> change of basis in the Simplex Method. In that way, we have a method where we can consider that there are outer iterations and inner iterations:

an <u>inner</u> iteration consists in solving, by the Decomposition Principle, a Restricted Program ($m+1$ unknowns and m equations);

an <u>outer</u> iteration consists, each time we have an extreme point x^ν of M, at the end of a series of inner iterations, in choosing an adequate new Restricted Program and in solving it by the Decomposition Principle.

The result of one outer iteration actually is one iteration of the Simplex Method, as applied to Program 1. But the number of inner iterations may be very large. As an illustrative example, let us take a Program which is solved by the Simplex Method in one iteration. Let us assume that there are 8 unknowns, that $p = 2$, $q = 5$, and that the five equations $Bx = b$ are such that the 3-dimensional intersection of $Bx = b$ with the positive orthant $x \geq 0$ is a

hexagonal prism, say Q, shown in Figure 3. The intersection with Q, of the linear variety $Ax = a$ (2 equations) is a segment, say M. Starting with an extremity, say x^o, we shall reach the other extremity, say $\bar{x}$, by one outer iteration; however, this might mean 6 inner iterations, which are listed below (at each iteration, the polyhedron Q is expressed with the first of three edges listed):

Start	$X^o\ X^1\ X^2$
1st iteration	$X^3\ X^1\ X^2$
2nd iteration	$X^4\ X^1\ X^2$
3rd iteration	$X^4\ X^1\ X^5$
4th iteration	$X^3\ X^1\ X^5$
5th iteration	$X^7\ X^1\ X^5$
6th iteration	$X^7\ X^8\ X^5$ END.

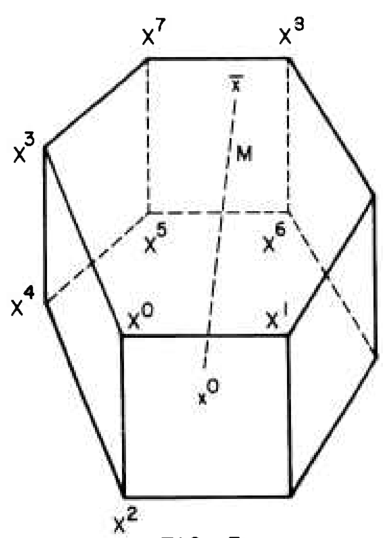

FIG. 3

3.2 Let us return to Step 3 of the Algorithm (Section 2). Once y^s has been computed, the set of x's such that

(1) $$x = x^o + \theta y^s$$

is, for $0 \leq \theta \leq \bar{\theta}$, an edge of M ; the maximum value of θ is easily computed as:

(2) $$\bar{\theta} = \min_{j \in J(x^o)} \left\{ \frac{x_j^o}{-y_j^s} \mid y_j^s < 0 \right\} ;$$

the point

(3) $$\bar{x} = x^o + \bar{\theta}\, y^s$$

the second extremity (other than x^o) of M on the considered edge. Now, the corresponding p-dimensional face, say $F^p(\bar{x})$, of Q (uniquely determined, since no degeneracy is allowed, by the property that $\bar{x}$ lies in its relative interior) has, as its p-dimensional variety support, the smallest dimensional variety which contains $F^p(\bar{x})$, that whose equations are

(4) $$B^{K'}\, x_{K'} = b$$

(5) $$x_{\bar{K}'} = 0 \quad \text{and} \quad x_{\bar{K}'} = 0$$

where

(6) $$K' = J(\bar{x}) = \{j : \bar{x}_j > 0\} .$$

It only remains to find $p+1$ vertices of $F^p(\bar{x})$ such that $\bar{x}$ belongs to their convex hull; this usually requires, if p is not a very small number, a lot of simplex-like iterations on the sub-subproblem (4) (merely to find in that manner a first edge of $F^p(\bar{x})$ needs p

iterations).

Then, unless p is very small, or unless the sub-subproblem (4) is so special that it is a simple matter to express $\bar{x}$ as a convex combination of vertices of $F^p(\bar{x})$, this second way to handle the Decomposition Principle in order to have extreme points of the original program seems unsatisfactory.

Another unpleasant feature is that, though we have assumed that we know the inverse matrix α, this is of quite no use, since the columns $A(X^\ell - X^o)$ may all have changed on passing from x^o to $\bar{x}$.

We now state the following result (which we shall use later on) as a simple consequence of Theorem 2, and of the definition of Y^s and $\bar{x}$:

<u>THEOREM 3</u>

$$c\,\bar{x} < cx^o$$

if and only if

$$(c - u_o A)\, Y^s < 0$$

3.3 One can overcome, in part, the difficulties encountered in section 3.2 in the following way: once Q has been expressed with an extreme point, say X^o, of $F^p(x^o)$, one will express x^o as the sum of X^o and a positive linear combination of some extreme half-lines of $C(X^o)$. More precisely, x^o will be expressed as

$$(1) \qquad x^o = X^o + \sum_{j \in I} Y^j \lambda_j^o ,$$

where the scalars λ_j^o are strictly positive by the nondegeneracy hypothesis (the notations have been explained at the end of section 1 and beginning of section 2). Now, everything in Section 2 can be rephrased so as to cover

the algorithm thus obtained.

We shall call $C^p(X^o; x^o)$ the p-dimensional finite cone

(2) $$x = X^o + \sum_{j \in I} Y^j \lambda_j ,$$

where the scalars λ_j are arbitrary and such that

(3) $$\lambda_j \geq 0 \quad , \forall\ j \in I$$

This cone has X^o as its origin, and (once this has been assumed) is characterized (due to nondegeneracy) by the following property: it is <u>the</u> p-dimensional face of $C(X^o)$ which contains x^o ; or equivalently by the following one: it is the smallest (in terms of inclusion) face of $C(X^o)$ containing x^o; or equivalently by another one: it plays for the polyhedron $F^p(x^o)$ the same role as $C(X^o)$ for the polyhedron Q . Moreover, the Y^j's here play the role of the $(X^\ell - X^o)$'s in Section 2. Let us restate Lemmas 1 and 2 and Theorem 1:

<u>Lemma 3</u> - The set of vectors $(Y^j)_{j \in I}$ is an independent set.

<u>Lemma 4</u> - The matrix whose colums are $(AY^j)_{j \in I}$ is a square nonsingular matrix.

We shall call Y^I the matrix whose columns are $(Y^j)_{j \in I}$, and Δ_I the inverse matrix of AY^I , the square nonsingular matrix of Lemma 4.

<u>THEOREM 4</u>

The hyperplane containing $C^p(X^o;x^o)$ and parallel to $Ax = 0$ and $cx = 0$ has an equation:

$$(c - u_o A)\ x = (c - u_o A)\ X^o$$

where u_o is the unique solution of

$$(c - uA)\, Y^I = 0$$

that is, u_o is explicitly given by:

$$u_o = cY^I \Delta_I \ .$$

It is a simple matter to see that u_o in theorems 1, 2, 3, and in theorem 4 is the same.

Let s be any index not in K; then the half-line

$$x = X^o + Y^s \Theta$$

(where Θ is a nonnegative arbitrary scalar) is an extreme half-line of $C(X^o)$ which does not belong to $C^p(X^o;\, x^o)$; conversely, we have any of these extreme half-lines in allowing s to take any value in $\bar{K}$. We consider $C^{p+1}(X^o; x^o, Y^s)$, the $(p+1)$-dimensional face of $C^p(X^o)$ generated by $(Y^j)_{j \in I+s}$.

$\bar{x}$ of Sections 3.1 and 3.2 precisely is the extremity (other than x^o) of the segment $[x^o, \bar{x}]$, intersection of this $(p+1)$-dimensional cone $C^{p+1}(X^o;\, x^o,\, Y^s)$ with the $(n-p)$-dimensional polyhedron, say P :

$$P = \{x : Ax = a,\ x \geq 0\} \ .$$

We notice that theorem 3 above applies here, since every symbol has the same meaning.

Once we have $\bar{x}$, we now have just to find <u>one</u> extreme point of $F^p(\bar{x})$, which needs some simplex like computations in the sub subproblem.

$B^{K'} x_{K'} = b$, but nothing more; this finally is less work than for the method of section 3.2, since we do not need here the other p extreme points that were necessary.

But this new method is as unsatisfactory as the preceding one when we come to the computation of the new Δ_I ; the new matrix Y^I may be

a matrix completely different from the old one, and we may have to completely invert the new matrix AY^I ; this will need p iterations of the method used in Section 2, Step 5, case 2 .

Although the algorithm thus obtained for passing from x^o to $\bar{x}$, needs less work than the preceding one, yet it is approximately p times more lengthy than another one to be described in Section 4.

3.4 There exists another procedure for finding a first vertex of $F^p(\bar{x})$: it simply consists in solving the following program, with X^o as a starting point:

$$\begin{aligned} &\text{Minimize } x_r \\ &\text{subject to} \\ &\qquad Bx = b \\ &\qquad x \geq 0, \end{aligned}$$

where r is the index j in formula (2) of section 3.2 such that

$$\bar{\theta} = \frac{x_r^o}{-y_r^s} \quad , \; y_r^s < 0 \; ,$$

that is the index $r \epsilon J(x^o)$ such that $\bar{x}_r = 0$.

This may lead to a fewer number of simplex computations in the sub-problem $Bx = b$, but do not overcome the other difficulties.

4. An Algorithm for Decomposition

We shall now turn our attention to the method outlined in Section 3.3, but will not restrict ourselves to feasible extreme points of Q . More precisely we shall say that a point X^o , belonging to the linear variety $Bx = b$, is a Q-primordial point if the following property holds:

Letting,

(1) $J = J(X^o) = \{j : X^o_j \neq 0\}$

the set of vectors $(B^j)_{j \in J(X^o)}$ is an independent set. We shall say that X^o is a nondegenerate primordial point for Q if $J(X^o)$ has exactly q elements. We shall assume throughout the sequel, that there is no degeneracy of any kind, including that described just now. But contrary to the simplifying assumption made in section 1 to 3, we do not assume any boundedness of the various polyhedra we will c nsider.

For any Q-primordial point X^o, one can repeat the considerations of Section 1, except for the slight change in the definition of $J(X^o)$ which has been given above. We define $C(X^o)$ as the smallest cone (in terms of inclusion) with apex X^o, and containing Q. Figure 4, as an example, assumes that Q is a 2-dimensional pentagon; a Q-primordial X^o is represented, together with two generators of $C(X^o)$ (this latter cone is an angle with apex X^o). Finally, the relations which define $C(X^o)$ are the same as in Section 1, that is

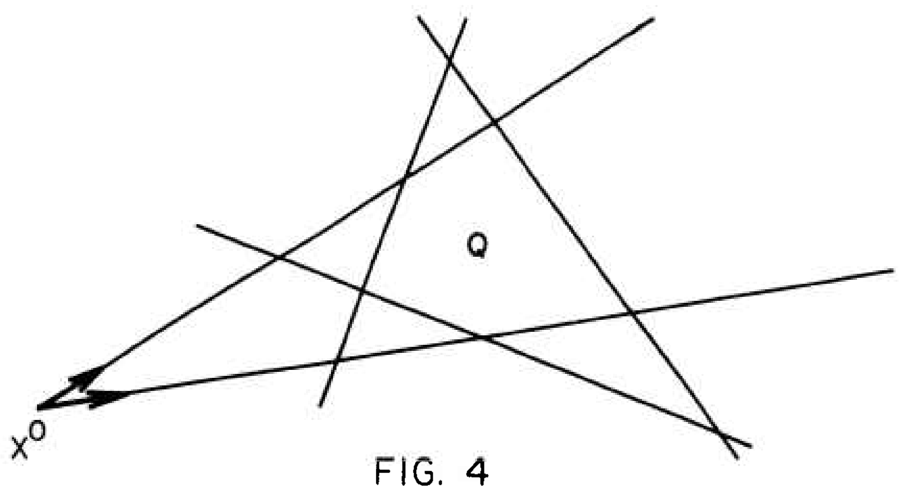

FIG. 4

(2) $By = 0$

(3) $y_{\bar{J}} \geq 0$ and y_J (unrestricted)

The characterization we gave in Section 1 for $C(X^o)$ as well as for the extreme half lines of this cone, still holds exactly, as well as the explicit formula given at the end of Section 1.

As in Section 2, let x^o be an extreme point of M, and define

$$K = J(x^o) = \{j : x^o_j > 0\}$$
$$I = J(x^o) - J(X^o) = K - J .$$

Then the unique p-dimensional face, say $C^p(X^o; x^o)$, of $C(X^o)$, containing x^o is the set of all the x's such that:

(4) $$x = X^o + \sum_{j \in I} Y^j \lambda_j ,$$

(5) $$\lambda_j \geq 0 , \forall_j \in I$$

In particular, we have

(6) $$x^o = X^o + \sum_{j \in I} Y^j \lambda^o_j ,$$

for some set of scalars $(\lambda^o_j)_{j \in I}$ satisfying (nondegeneracy) ;

(7) $$\lambda^o_j > 0 \quad , \quad \forall_{j \in I}$$

The extreme half-lines of $C(X^o)$ which are not in $C^p(X^o; x^o)$ are

(8) $$X = X^o + \theta Y^s ,$$

where the scalar θ is ≥ 0 , and $s \in \bar{K}$.

Lemma 3 and 4 and Theorem 4 of the preceding section still hold exactly.

We shall give an interpretation of u_o . Let us consider the dual inequalities to Program 1 :

$$uA + vB \leq c$$

where u , v respectively are p-dimensional and q-dimensional row vectors. Let u'_o , v_o be the dual prices associated with the basic solution x^o :

(u'_o , v_o) is the unique (u, v) such that

$$u'_o \, A^k + v_o \, B^k = c^k \ , \ \forall \, k \in K \ .$$

We notice now that if $k \in \bar{K}$ and $j \in I$, then $Y^j_k = 0$ (see the definitions of K, I, Y^j). From the preceding inequalities we have then

$$u'_o \, AY^j + v_o \, BY^j = cY^j \ , \ \forall \, j \in I \ .$$

But $BY^j = 0$, $j \in I$ (see definition of Y^j) .

We conclude

$$u'_o \, AY^I = cY^I \quad , \qquad u'_o = cY^I \Delta_I$$

which (by theorem 4) shows that $u'_o = u_o$.

Thus the p-dimensional row-vector u_o in theorem 4 is the dual price-vector associated with the set of equations $Ax = a$ and with the basic solution x^o of Program 1.

As the same kind of proof applies to the row-vector u_o of Sections 2 and 3, we see that the u_o's in all consideration of this paper are the same.

THEOREM 5

The (p+1)-dimensional cone $C^{p+1}(X^o;\, x^o,\, Y^s)$ defined by

(9) $$x = X^o + \sum_{j\in I} Y^j \lambda_j + Y^s\,\theta\ ,$$

(10) $$\lambda_j \geq 0\ ,\ \forall\ j \in I\ ,$$

(11) $$\theta \geq 0$$

intersects the p-dimensional polyhedron $P(Ax = a\ ,\ x \geq 0)$ on a half-line or a segment, one extremity of which is x^o .

We shall first compute the intersection of the smallest linear variety containing $C^{p+1}(X^o;\, x^o,\, Y^s)$ with the linear vzriety $Ax = a$; this intersection is the line defined by

(12) $$x = X^o + Y^I\,\lambda_I + Y^s\,\theta$$

(13) $$Ax = a\ ,$$

where λ_I is the p-dimensional vector whose components are $(\lambda_j)_{j\in I}$.
A substitution gives ,

$$AY^I\lambda_I = a - AX^o - AY^s\theta\ ,$$

hence, by multiplying on the left by Δ_I , the inverse matrix of AY^I:

(14) $$\lambda_I = \Delta_I\,(a - AX^o) - \Delta_I\,(AY^s)\,\theta\ .$$

The intersection is then the line

$$x = x^o + (Y^s - Y^I\,\Delta_I\,AY^s)\,\theta\ ,$$

that is

(15) $$x = x^o + \bar{y}\,\theta\ ,$$

setting

(16) $\quad \bar{y} = Y^s - Y^I \Delta_I AY^s$.

To impose $x \in M$ and since $x^o \in M$ it follows that $A\bar{y} = 0$ and

(17) $\quad x^o + \bar{y}\,\theta \geq 0 \ , \ \theta \geq 0$.

If $\bar{y} \geq 0$, then the half-line (15) is contained in M ; if not, then there is a maximum value of θ in (17), say $\bar{\theta}$. The point

$$\bar{x} = x^o + \bar{y}\,\bar{\theta}$$

is then the vertex of M , other than x^o , on the edge of M defined by the conditions

(18) $\quad Ax = a$

(19) $\quad Bx = b$

(20) $\quad x_j \geq 0 \ , \quad \forall \ j \in K+s$

(21) $\quad x_j = 0 \ , \quad \forall \ j \in \overline{K+s}$

In any case the following theorems will be useful:

THEOREM 6

$$c\bar{y} < 0$$

if and only if

$$(c - u_o A)\, Y^s < 0 \ .$$

PROOF:

$$\begin{aligned} c\bar{y} &= (c - cY^I \Delta_I A)\, Y^s \quad , \quad \text{by (16)} \\ &= (c - u_o A)\, Y^s \quad , \quad \text{by theorem 4.} \end{aligned}$$

THEOREM 7

x^o is an optimal solution to Program 1 if and only if

$$(c - u_o A)\, Y^j \geq 0 \quad , \forall\ j \in \overline{K}$$

PROOF:

x^o is optimal if and only if for any edge of M out of x^o

$$x = x^o + \theta\, \bar{y}, \quad 0 \leq \theta \leq \bar{\theta} ,$$

we have:

$$c\bar{y} \geq 0$$

It only remains to notice that <u>any</u> edge of M, out of x^o is uniquely associated with a Y^s by (16), as it has been shown in (18) to (21).

We shall now give the outline of the Algorithm. Starting with x^o, X^o, Δ_I, and the expression of Q with X^o (which automatically gives any Y^j, $j \in \overline{J}$, we want) we successively compute:

1, $\quad u_o = cY^I \Delta_I$

2.1, If $(c - u_o A)Y^j \geq 0$, $\forall\ j \in \overline{K}$, then x^o is an optimal solution of Program 1;

2.2, if not, let $s \in \overline{K}$ be such that

$$(c - u_o A)\, Y^s < 0 ;$$

3, compute

$$\bar{y} = Y^s - Y^I \Delta_I\, AY^s$$

3.1, if $\bar{y} \geq 0$, then the minimum of cx in Program 1 is $-\infty$;

3.2, if not, compute

$$\bar{\Theta} = \frac{x_r^o}{-\bar{y}_r} = \min_{j \in K} \left\{ \frac{x_j^o}{-\bar{y}_j} \mid \bar{y}_j < 0 \right\}$$

4, We already have $\bar{x}$ (the new x^o from 17). We must compute a new X^o, a new Δ_I, a new expression for Q in terms of the new X^o. The following section is devoted to this Step 4 (last step of the finitely iterative algorithm).

5. Discussion and Formulas for the Last Step of the Iteration

We shall use the formulas

(1) $$\bar{x} = X^o + \sum_{j \in I} Y^j \bar{\lambda}_j + Y^s \theta$$

(2) $$\bar{\lambda}_j \geq 0 \quad , \quad \forall_j \in I$$
$$\bar{\Theta} > 0$$

We distinguish three (nondegenerate, exhaustive) cases respectively illustrated in Figs. 5, 6, 7.

Case 1. X^o belongs to the hyperplane $x_r = 0$.

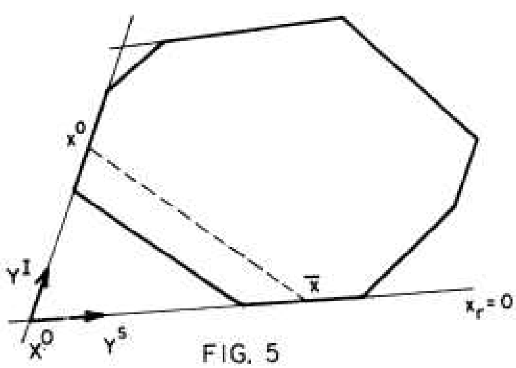

FIG. 5

Then $\bar{x}$ is a point of the cone $C^{p+1}(X^o; x^o, Y^s)$ which at the same time is on the boundary of this cone. This means that $\bar{x}$ belongs to one of its p-dimensional face, namely the face obtained by setting $x_r = 0$. This shows that $r \in I$ and $\bar{\lambda}_r = 0$.

Conversely, if $r \in I$ and $\bar{\lambda}_r = 0$, then (1) gives $\bar{x}_r = 0$.

In that case, we set $\bar{X}$ (the new X^o)

$$\bar{X} = X^o :$$

i.e., we do not make any change of basis in the subproblem $Bx = b$ (Q remains expressed with X^o) ;

I becomes $I - r + s$;

J remains as it stands.

The computation of the new Δ_I, say $\bar{\Delta}$, easy, using the Sherman-Morrison formula. In fact, the column AY^r of the matrix AY^I (of which Δ_I is the inverse) is changed into AY^s, and that is all the change to be made:

$$(3) \qquad \bar{\Delta} = \Delta_I + \tau \Delta_I A (Y^r - Y^s) \Delta_r ,$$

where

$$(4) \qquad \tau = \frac{1}{1-\Delta_r A(Y^r - Y^s)} = \frac{1}{\Delta_r AY^s} ,$$

and Δ_r is the rth row of Δ_I .

Notice that in (3) $\Delta_I \; AY^r$ is simply the rth unit vector.

Case 2. The hyperplane $x_r = 0$ intersects the line $x = X^O + \theta Y^s$

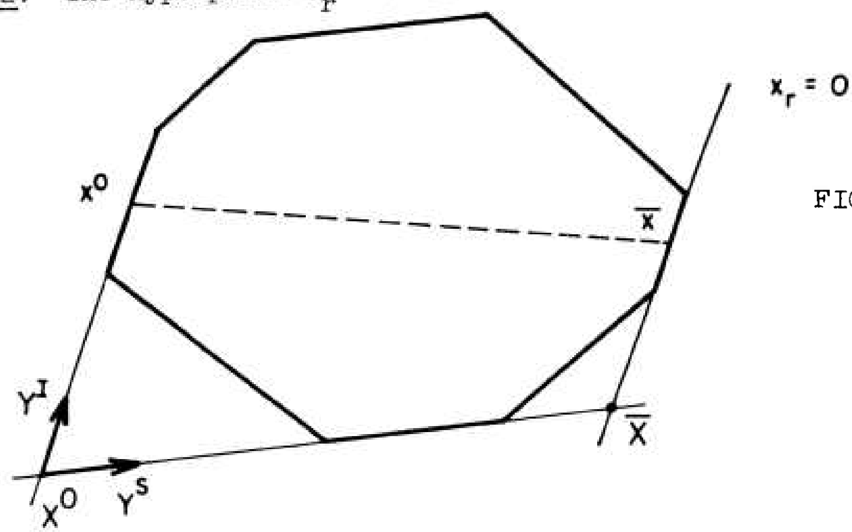

FIG. 6

(where θ is not restricted, notice that in Case 1, this line is contained in the hyperplane $x_r = 0$).

In the present case, we take $\overline{X}$ (the new X^O) as the intersection. r does not belong to I (otherwise, we should be in case 1), then $r \in J$. Then

I stands at it is;

J becomes $J - r+s$ (one change of basis in the subproblem, with $\tilde{B}^s_r$ as a pivotal element).

We again have to compute the new Δ_I, say $\overline{\Delta}$. First, the expression of Y^I has changed, from Y^I to

(5) $$Y^I + \frac{1}{Y^s_r}(e^r - Y^s)\, Y^I_r$$

(where Y^I_r is the rth row of the matrix Y^I, and e^r is the rth unit column-vector).

Then AY^I has changed to

(6) $$AY^I + \frac{1}{Y^s_r}(A^r - AY^s)\, Y^I_r \;;$$

Application of the Sherman-Morrison formula gives:

(7) $$\bar{\Delta} = \Delta_I - \tau\Delta_I(A^r - AY^s)Y_r^I\Delta_I ,$$

(8) $$\tau = \frac{1}{Y_r^s + Y_r^I\Delta_I(A^r - AY^s)}$$

<u>Case 3</u>. The hyperplane $x_r = 0$ is parallel to the line $x = X^o + \theta Y^s$

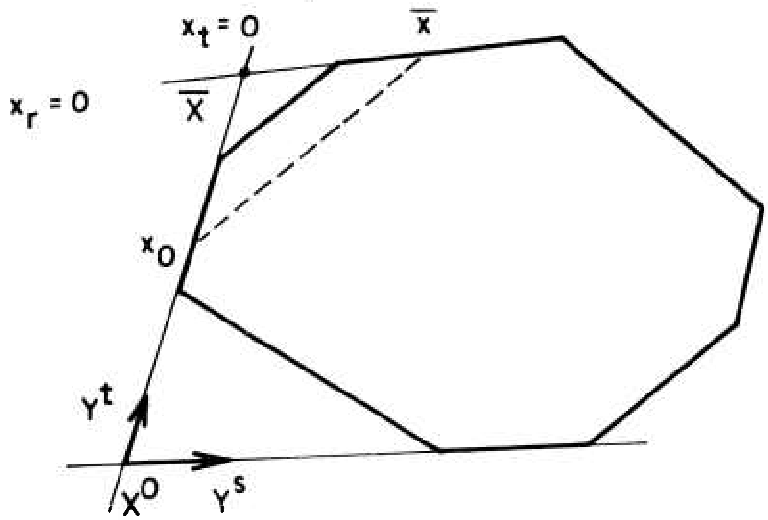

FIG. 7

(this is not a case of degeneracy, since it cannot be avoided by an ε-perturbation of the cost function or the right-hand side). There must be an extreme half-line of $C^p(X^o ; x^o)$ which intersects the hyperplane $x_r = 0$ (if not $C^{p+1}(X^o ; x^o , Y^s)$ should be parallel to this hyperplane; but this is not the case, since $\bar{x}$ lies in the intersection). This shows that there exists some $\tilde{B}_r^t \neq 0$, with $t \in I$.

Once t is thus chosen in I , we make a succession of two substeps:

<u>Substep 1</u>. Apply case 2 for changing J into $J - R + t$;

<u>Substep 2</u>. Apply case 1 for changing I into $I - t + s$.

6. <u>An Alternative Approach to the Same Algorithm</u>

We shall not give a general idea by which one can find, in addition to the algorithm above, the algorithm of G. B Dantzig, A. S. Manne, W. Orchard-Hays, and J. T. Robacher, as well as some others.

We apply the simplex method to Program 1. Let x^o be a basic feasible (nondegenerate) solution; let K be the set.

$$K = \{j : x_j^o > 0\}$$

Let J be a subset of K such that the matrix, say B^J, whose columns are $(B^j)_{j \in J}$ is a square nonsingular matrix, and let I be $K - J$, the complement of J with respect to K. We call

A^I the matrix whose columns are (A^j) $j \in I$

B^I the matrix whose columns are (B^j) $j \in I$

A^J the matrix whose columns are (A^j) $j \in J$.

We shall take, as an example, the "full inverse" method. The inverse of the basic matrix is

$$(1) \qquad \begin{pmatrix} A^I & A^J \\ B^I & B^J \end{pmatrix}^{-1} = \begin{pmatrix} \Delta_I & -\Delta_I A^J \beta_J \\ -\beta_J B^I \Delta_I & \beta_J + \beta_J B^I \Delta_I A^J \beta_J \end{pmatrix}$$

where

$$(2) \qquad \beta_J = (B^J)^{-1}$$

$$(3) \qquad \Delta_I = (A^I - A^J \beta_J B^I)^{-1} .$$

First we notice that, with the notations of Section 5:

$$Y_I^I = I \times I \text{ identity matrix}$$

$$Y_J^I = \bar{\bar{B}}^I = -\beta_J B^I ,$$

$$Y_{\bar{K}}^I = 0 ,$$

so that

(4) $$AY^I = A^I - A^J\beta_J B^I ;$$

then the Δ_I under consideration here is the same as in Section 4 and 5.

Second, in the simplex method we have to compute

(5) $$\delta^j = c^j - u_o A^j - v_o B^j$$

for every $j \in \overline{K}$, where u_o, v_o are given, using (1) and setting

$$c^I = (c^i)_{i \in I}$$
$$c^J = (c^j)_{j \in J} ,$$

by

(6) $$u_o = c^I \Delta_I - c^J \beta_J B^I \Delta_I$$

(7) $$v_o = c^J \beta_J + c^J \beta_J B^I \Delta_I A^J \beta_J - c^I \Delta_I A^J \beta_J .$$

Formula (6) is exactly

(8) $$u_o = cY^I \Delta_I ,$$

which is the method used in Section 4 for computing u_o, and formula (7) may be written

(9) $$v_o = c^J \beta_J - cY^I \Delta_I A^J \beta_J ,$$

whence

(10) $$\delta^s = c^s - u_o A^s - (c^J \beta_J - cY^I \Delta_I A^J \beta_J) B^s ,$$
$$\delta^s = c^s - c^J \beta_J B^s - cY^I \Delta_I (A^s - A^J \beta_J B^s) ,$$

and finally

$$\delta^{s} = c\ (Y^{s} - Y^{I}\Delta_{I}AY^{s})\ ,$$

that is

(11) $$\delta^{s} = c\bar{y}\ ;$$

but δ^{s} can also be written from (10) :

$$\delta^{s} = c^{s} - u_{o}\ A^{s} - (c^{J} - u_{o}\ A^{J})\ \beta_{J}B^{s}\ ,$$

that is

(12) $$\delta^{s} = (c - u_{o}A)\ Y^{s}\ .$$

Formulas (12) and (11) gives an alternative explanation of steps 2 and 3 of the algorithm of Section . Moreover, the "updated column s" , in Program 1 is the vector.

$$\begin{pmatrix} \Delta_{I}\ A^{s} - \Delta_{I}\ A^{J}\ \beta_{J}\ B^{s} \\ -\beta_{J}B^{I}\ \Delta_{I}A^{s} + \beta_{J}B^{s} + \beta_{J}B^{I}\ \Delta_{I}A^{J}\ \beta_{J}B^{s} \end{pmatrix}$$

that is

$$\begin{pmatrix} \Delta_{I}\ AY^{s} \\ -Y_{J}^{s} + Y_{J}^{I}\Delta_{I}AY^{s} \end{pmatrix}$$

which is easily reduced to $-\bar{y}_{K}$, the opposite of $\bar{y}_{K}$ in step 3 ($\bar{y}_{K}$ being the vector $(\bar{y}_{j})_{j\in K}$) .

As for step 4, it simply is the updating of Δ_{I} . (explained in Section 5) .

Finally, the algorithm of Sections 4 and 5 is nothing else than the adaptation of the Simplex Method if we assumed that we keep, in the memory of the computer:

$$c\ ,\ A\ ,\ x^{o}\ ,\ \Delta_{I}\ ,\ \beta_{J}B\ ,$$

so that at the end of the iteration, we have to compute:

the new x^{o} (which is easy)

the new Δ_{I} ,

the new $\beta_{J}B$ (change of basis in the subproblem) .

As a slight modification, one can keep in the memory:

$$c\ ,\ A\ ,\ B\ ,\ x^{o}\ ,\ \Delta_{I}\ ,\ \beta_{J}\ .$$

The method thus obtained only differs from our algorithm in that now we use the full inverse for the subproblem, whereas in our algorithm we use the Simplex Tableau; the algorithm of Sections 4 and 5 is then a very slight modification of the algorithm of G. B. Dantzig, A. S. Manne, W. Orchard-Hays, and J. T. Robacker.

It is clear that, in varying the quantities kept in the memory, one can have, following the same line of approach, many other modifications of the simplex method which take into account the horizontal splitting of the matrix of Program 1.

APPENDIX 1

The Sherman-Morrison's Formula [5].

Let A be an $n \times n$ regular matrix, and consider the following modification of A :

$$A \longrightarrow A + BC$$

where B is an $n \times m$ and C an $m \times n$ matrix (as a particular case, which actually is the only one we needed in the paper, B is an n-dimensional column-vector and C is an n-dimensional row-vector). There is, accordingly, a modification of the inverse of A :

$$A^{-1} \longrightarrow A^{-1} - A^{-1} BM^{-1} CA^{-1} ,$$

where, denoting by I the $m \times m$ identity matrix,

$$M = I + CA^{-1} B$$

(in the previous particular case M is a scalar).

PROOF: it suffices to check by direct computation that

$$(A^{-1} - A^{-1} BM^{-1} CA^{-1}) (A + BC)$$

is equal to the $n \times n$ identity matrix.

APPENDIX 2

On Notations

We have adopted the following systematic notations for submatrices of a given matrix [2] :

Let A be a matrix, G be a subset of the set of rows indices of A , and H be a subset of the set of columns indices of A.

We denote by A_i the ith row of A , by A^j the jth column of A , by A_i^j the element at the intersection of the ith row and jth column of A . Moreover:

A_G is the submatrix of A whose rows are $(A_i)_{i \in G}$,

A^H is the submatrix of A whose columns are $(A^j)_{j \in H}$,

A_G^H is the submatrix of A whose elements are $(A_i^j)_{(i,j) \in G x H}$.

In particular, this applies to vectors, so that if x is a column-vector and G a subset of its components indices, then x_G is the sub-column vector whose components are $(x_i)_{i \in G}$; likewise, if u is a row-vector and H a subset of its components indices, then u^H is the sub-row vector whose components are $(j^j)_{j \in H}$.

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